UG-FT-164/04
CAFPE-34/04

FORMULAE FOR A NUMERICAL COMPUTATION OF ONE-LOOP TENSOR INTEGRALS [a]

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A numerical and iterative approach for computing one-loop tensor integrals is presented.

1 Introduction

In Reference [1] a new approach has been introduced for computing, recursively and numerically, one-loop tensor integrals. Here we describe a few modifications of the original method that allow a more efficient numerical implementation of the algorithm. We keep all of our notations as in Ref. [1] and, in particular, put a bar over n-dimensional quantities and a tilde over ϵ-dimensional objects ($n = 4 + \epsilon$). The formulae we want to modify are Eqs. (9), (35), (48), (52) and (54) of Ref. [1], that, all together, allow to reduce any $(m+1)$-point tensor integral with $m \geq 2$ to the standard set of scalar one-loop functions [2].

Such formulae are not symmetric when interchanging any pair of loop denominators $\bar{D}_k$, because are derived under the assumption that at least one of them (identified with $\bar{D}_0$) carries a vanishing external momentum, namely

$$\bar{D}_k = (\bar{q} + p_k)^2 - m_k^2\,, \quad k = 0, \cdots, m\,, \quad p_0^\mu = 0\,. \tag{1}$$

Already after the first iteration, terms appear in which the denominator $\bar{D}_0$ is canceled by a $\bar{D}_0$ reconstructed in the numerator, so that the resulting integrals do not fulfill any longer the assumption of Eq. (1). A shift of the integration variable $\bar{q}$ is then needed to bring them back to a form suitable to apply the algorithm again. However, shifting $\bar{q}$ may generate a large amount of terms, especially when dealing with high rank tensors, so that deriving more symmetric formulae, in which $p_0^\mu \neq 0$, is clearly preferable.

[a] Talk given at LCWS2004, Paris, France, April 2004.
Work supported by the EU under contract HPRN-CT-2002-00149 and by MECD under contract SAB2002-0207.

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A second useful modification is related to the problem outlined in Sec. 6 of Ref. [1], that occur when $p_1^2 = 0$ and $p_2^2 \neq 0$ ($p_1^2 \neq 0$ and $p_2^2 = 0$) and $(p_1 \cdot p_2) \sim 0$. For those kinematical configurations a new linear combination of the momenta p_1 and p_2 is needed as a basis of the reduction to ensure the numerical stability of the algorithm. Once again, it is better to include such cases right from the beginning.

2 The General Recursion Formula

When $p_0^\mu \neq 0$, the n-dimensional version of Eq. (9) of Ref. [1] should be modified as follows

$$\begin{aligned}
I^{(n)}_{m;\,\mu\nu\rho\cdots\tau} &= \frac{\beta}{2\gamma}\,T_{\mu\nu\lambda\sigma}\,\left\{J^{(n)\,\lambda\sigma}_{m;\,\rho\cdots\tau}\right\} \\
&- \frac{1}{4\,\gamma}T_{\mu\nu}\left\{(m_0^2-p_0^2)\,I^{(n)}_{m;\,\rho\cdots\tau} + I^{(n)}_{m-1;\,\rho\cdots\tau}(0) - 2\,p_{0\alpha}I^{(n)\,\alpha}_{m;\,\rho\cdots\tau} - I^{(n;\,2)}_{m;\,\rho\cdots\tau}\right\} \\
&- \frac{1}{4\,\gamma}T_{\mu\nu\lambda}\left\{h_3 I^{(n)\,\lambda}_{m;\,\rho\cdots\tau} + I^{(n)\,\lambda}_{m-1;\,\rho\cdots\tau}(3) - I^{(n)\,\lambda}_{m-1;\,\rho\cdots\tau}(0) - \frac{2\beta}{\gamma}k_{3\alpha}\,J^{(n)\,\alpha\lambda}_{m;\,\rho\cdots\tau}\right\}\,,
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
J^{(n)\,\lambda\sigma}_{m;\,\rho\cdots\tau} &= (h_1 r_2^\lambda + h_2 r_1^\lambda)\,I^{(n)\,\sigma}_{m;\,\rho\cdots\tau} + (r_2^\lambda + \xi_2 r_1^\lambda)\,I^{(n)\,\sigma}_{m-1;\,\rho\cdots\tau}(1) \\
&+ (r_1^\lambda + \xi_1 r_2^\lambda)\,I^{(n)\,\sigma}_{m-1;\,\rho\cdots\tau}(2) - [r_1^\lambda(1+\xi_2) + r_2^\lambda(1+\xi_1)]\,I^{(n)\,\sigma}_{m-1;\,\rho\cdots\tau}(0)\,.
\end{aligned} \tag{3}$$

and where the extra integrals $I^{(n;\,2)}_{m;\,\rho\cdots\tau}$ are defined in Eq. (77) of Ref. [1].

In the previous Equations, $k_i = p_i - p_0$ and the massless 4-momenta $\ell_{1,2}$ to be used as a basis of the reduction algorithm, as in Eq. (13) of Ref. [1], are such that

$$s_1 = \ell_1 + \alpha_1 \ell_2\,, \quad s_2 = \ell_2 + \alpha_2 \ell_1\,, \tag{4}$$

where $s_{1,2}$ are suitable linear combinations of $k_{1,2}$

$$s_1 = k_1 + \xi_1 k_2\,, \quad s_2 = k_2 + \xi_2 k_1\,. \tag{5}$$

By choosing

$$\xi_1 = \frac{1}{2}\,\mathrm{sign}(k_2^2)\,\mathrm{sign}(k_1\cdot k_2) \quad \text{and} \quad \xi_2 = \frac{1}{2}\,\mathrm{sign}(k_1^2)\,\mathrm{sign}(k_1\cdot k_2)\,, \tag{6}$$

the quantity

$$\gamma = \frac{s_1^2 s_2^2}{(s_1\cdot s_2) \mp \sqrt{\Delta}} \equiv (s_1\cdot s_2) \pm \sqrt{\Delta}\,, \quad \Delta = (s_1\cdot s_2)^2 - s_1^2 s_2^2\,, \tag{7}$$

defined in Eq. (62) of Ref. [1] only vanishes when $k_1^2 = k_2^2 = (k_1 \cdot k_2) = 0$, that always corresponds to collinear configurations cut away in physical observables, therefore solving the second problem outlined in the Introduction. The tensors $T_{\mu\nu\lambda\sigma}$, $T_{\mu\nu\lambda}$, $T_{\mu\nu}$ and the 4-vectors r_{12} are defined as in Ref. [1], but in terms of $\ell_{1,2}$ given in Eq. (4) and with the replacement $p_3 \to k_3$. Finally

$$\begin{aligned} h_1 &= (m_1^2 - p_1^2) + \xi_1 (m_2^2 - p_2^2) - (1+\xi_1)(m_0^2 - p_0^2)\,, \\ h_2 &= (m_2^2 - p_2^2) + \xi_2 (m_1^2 - p_1^2) - (1+\xi_2)(m_0^2 - p_0^2)\,, \\ h_3 &= (m_3^2 - p_3^2) - (m_0^2 - p_0^2)\,, \\ \frac{\beta}{\gamma} &= \pm \frac{1}{2\sqrt{\Delta}}\,. \end{aligned} \tag{8}$$

The derivation of Eq. (2) closely follows the derivation of Eq. (9) of Ref. [1]. For example, choosing $\ell_{1,2}$ as in Eq. (4), the quantity

$$D_\mu = \frac{1}{\beta}\left[2\,(q\cdot\ell_1)\ell_{2\mu} + 2\,(q\cdot\ell_2)\ell_{1\mu}\right] \tag{9}$$

defined in Eq. (18) of Ref. [1] can be rewritten as

$$\begin{aligned} D_\mu &= [\bar{D}_1 + \xi_1 \bar{D}_2 - (1+\xi_1)\bar{D}_0 + h_1]\, r_{2\mu} \\ &+ [\bar{D}_2 + \xi_2 \bar{D}_1 - (1+\xi_2)\bar{D}_0 + h_2]\, r_{1\mu}\,, \end{aligned} \tag{10}$$

and generates the term $J^{(n)}$ of Eq. (3). Analogously, choosing $b = k_3$ in Eq. (22) of Ref. [1], generates the first part of the last term of Eq. (2), because, when $p_0^\mu \neq 0$

$$2\,(q\cdot k_3) = \bar{D}_3 - \bar{D}_0 + h_3\,. \tag{11}$$

Finally, the equality

$$q^2 = \bar{D}_0 + (m_0^2 - p_0^2) - 2\,(q\cdot p_0) - \tilde{q}^2\,, \tag{12}$$

is the origin of the second row of Eq. (2).

3 Three-point Tensors

When $p_0^\mu \neq 0$, rank 2 and rank 3 three-point tensor integrals need a separate treatment. The relevant formulae follow by adapting the theorems in Eqs. (37) and (40) of Ref. [1] to the case $p_0^\mu \neq 0$:

$$\int d^n\bar{q}\,\frac{1}{\bar{D}_0\bar{D}_1\bar{D}_2}\,[(q+p_0)\cdot\ell_3]^i = 0\,,$$

$$\int d^n\bar{q}\,\frac{1}{\bar{D}_0\bar{D}_1\bar{D}_2}[(q+p_0)\cdot\ell_4]^i = 0\,, \qquad \forall\, i = 1,2,3\cdots \quad \text{and}$$
$$\int d^n\bar{q}\,\frac{1}{\bar{D}_0\bar{D}_1\bar{D}_2}[(q+p_0)\cdot\ell_{3,4}]^2\, q_\rho \;=\; 0\,. \tag{13}$$

The final results read as follows

$$\begin{aligned}
I^{(n)}_{2;\,\mu\nu} &= \frac{\beta}{2\gamma}\,T'_{\mu\nu\lambda\sigma}\,\left\{J^{(n)\,\lambda\sigma}_{2}\right\} \;-\; \frac{1}{4\,\gamma}t_{\mu\nu}\left\{(m_0^2-p_0^2)\,I^{(n)}_2 + I^{(n)}_1(0)\right.\\
&- \left. 2\,p_{0\alpha}I^{(n)\,\alpha}_2 - I^{(n;\,2)}_2\right\} \;-\; \frac{1}{4\,\gamma}T'_{\mu\nu\lambda}\left\{p_0^\lambda\, I^{(n)}_2\right\}\,,\\
I^{(n)}_{2;\,\mu\nu\rho} &= \frac{\beta}{2\gamma}\,T'_{\mu\nu\lambda\sigma}\,\left\{J^{(n)\,\lambda\sigma}_{2;\,\rho}\right\} \;-\; \frac{1}{4\,\gamma}t_{\mu\nu}\left\{(m_0^2-p_0^2)\,I^{(n)}_{2;\,\rho} + I^{(n)}_{1;\,\rho}(0)\right.\\
&- \left. 2\,p_{0\alpha}I^{(n)\,\alpha}_{2;\,\rho} - I^{(n;\,2)}_{2;\,\rho}\right\} \;-\; \frac{1}{4\,\gamma}T'_{\mu\nu\lambda}\left\{-p_0^\lambda\, I^{(n)}_{2;\,\rho} - 2\,I^{(n)\,\lambda}_{2;\,\rho}\right\}.
\end{aligned} \tag{14}$$

$J^{(n)}$ is given in Eq. (3) and

$$\begin{aligned}
t_{\mu\nu} &= \ell_{3\mu}\ell_{4\nu} + \ell_{4\mu}\ell_{3\nu}\,,\\
T'_{\mu\nu\lambda} &= -\frac{\ell_{3\mu}\ell_{3\nu}\ell_{4\lambda}(p_0\cdot\ell_4) + \ell_{4\mu}\ell_{4\nu}\ell_{3\lambda}(p_0\cdot\ell_3)}{\gamma}\,.
\end{aligned} \tag{15}$$

4 Rank One Tensors

In this section we adapt Eqs. (48), (52) and (54) of Ref. [1] to the case $p_0^\mu \neq 0$.

4.1 The $m=2$ case

With $t_{\alpha\mu}$ defined in Eq. (15) we get

$$I^{(n)}_{2;\,\mu} = \frac{\beta}{\gamma}J^{(n)}_{2;\,\mu} + \frac{p_0^\alpha}{2\gamma}\,t_{\alpha\mu}\,I^{(n)}_2\,. \tag{16}$$

4.2 The $m=3$ case

With $T_{\mu\nu\lambda}$ defined as in Eq. (23) of Ref. [1] we get

$$\begin{aligned}
I^{(n)}_{3;\,\mu} &= \frac{\beta}{\gamma}\,J^{(n)}_{3;\,\mu} \;+\; \frac{1}{4}\left[\frac{\ell_{3\mu}}{(k_3\cdot\ell_3)} + \frac{\ell_{4\mu}}{(k_3\cdot\ell_4)}\right]\\
&\times\; \left\{h_3 I^{(n)}_3 + I^{(n)}_2(3) - I^{(n)}_2(0) - \frac{2\beta}{\gamma}k_3^\lambda\,J^{(n)}_{3;\,\lambda}\right\}\\
&-\; \frac{1}{4\gamma}\,T_{\mu\nu\lambda}\,(p_0^\nu k_3^\lambda - p_0^\lambda k_3^\nu)\,I^{(n)}_3\,.
\end{aligned} \tag{17}$$

4.3 The $m > 3$ case

The generalization of Eq. (54) of Ref. [1] reads

$$
\begin{aligned}
I^{(n)}_{m;\,\mu} &= \frac{\beta}{\gamma} J^{(n)}_{m;\,\mu} + \frac{\ell_{3\mu}\ell_{4\alpha} - \ell_{3\alpha}\ell_{4\mu}}{2\,\delta} \\
&\times \left\{ k_3^\alpha \left[h_4 I^{(n)}_m + I^{(n)}_{m-1}(4) - I^{(n)}_{m-1}(0) - \frac{2\beta}{\gamma} k_{4\lambda}\, J^{(n)\,\lambda}_m \right] \right. \\
&- \left. k_4^\alpha \left[h_3 I^{(n)}_m + I^{(n)}_{m-1}(3) - I^{(n)}_{m-1}(0) - \frac{2\beta}{\gamma} k_{3\lambda}\, J^{(n)\,\lambda}_m \right] \right\}, \qquad (18)
\end{aligned}
$$

where $\delta = (\ell_3 \cdot k_4)(\ell_4 \cdot k_3) - (\ell_3 \cdot k_3)(\ell_4 \cdot k_4)$, and $h_4 = (m_4^2 - p_4^2) - (m_0^2 - p_0^2)$.

5 The Extra Integrals

In most practical cases, the extra integrals, such as $I^{(n;\,2)}_{m;\,\rho\cdots\tau}$ in Eq. (2), are either zero or scaleless, so that, even when $p_0^\mu \neq 0$, one can directly use the results given in Appendix B of Ref. [1]. In all other cases modifications are needed. For example, Eqs. (78) and (83) of Ref. [1] must be replaced by

$$
\begin{aligned}
I^{(n;\,2)}_{2;\,\mu} &= \frac{i\pi^2}{6}(p_{0\mu} + p_{1\mu} + p_{2\mu}) + \mathcal{O}(\epsilon)\,, \\
I^{(n;\,2)}_{1} &= -i\frac{\pi^2}{2}\left[m_1^2 + m_0^2 - \frac{(p_1 - p_0)^2}{3} \right] + \mathcal{O}(\epsilon)\,. \qquad (19)
\end{aligned}
$$

6 Conclusion

We have derived a set of formulae to efficiently implement the n-dimensional reduction algorithm presented in Ref. [1]. As for the three-point tensors, we limited our analysis to ranks ≤ 3. For higher ranks, a general formula can be easily derived, with the help of Appendix C of Ref. [1], using the theorems of Eq. (13).

References

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2. G. 't Hooft and M. J. G. Veltman, Nucl. Phys. B **153** (1979) 365.